ISRAEL JOURNAL OF MATHEMATICS **134** (2003), 307–315

ON THE SYMMETRIC SQUARE. UNSTABLE TWISTED CHARACTERS

BY

YUVAL Z. FLICKER

Department of Mathematics, The Ohio State University
231 W. 18th Ave., Columbus, OH 43210-1174, USA
e-mail: flicker@math.ohio-state.edu

AND

DMITRII ZINOVIEV

Institute for Problems of Information Transmission, Russian Academy of Sciences
Bolshoi Karetnyi per. 19, GSP-4, Moscow, 101447, Russia
email: zinov@iitp.ru

ABSTRACT

We provide a purely local computation of the (elliptic) twisted (by "transpose-inverse") character of the representation $\pi = I(\mathbf{1})$ of PGL(3) over a p-adic field induced from the trivial representation of the maximal parabolic subgroup. This computation is independent of the theory of the symmetric square lifting of [IV] of automorphic and admissible representations of SL(2) to PGL(3). It leads — see [FK] — to a proof of the (unstable) fundamental lemma in the theory of the symmetric square lifting, namely that corresponding spherical functions (on PGL(2) and PGL(3)) are matching: they have matching orbital integrals. The new case in [FK] is the unstable one. A direct local proof of the fundamental lemma is given in [V].

This work continues the paper [FK], whose notations we use. Our aim is to prove Proposition 1 of [FK] without using Theorem 0 there. Namely we provide a purely local computation of the twisted character of $\pi = I(\mathbf{1})$. Our model of π is that of [FK], where the twisted character χ_π is computed directly and locally but only for the anisotropic twisted conjugacy class δ' (see [FK], proof of

Received January 18, 2002

Proposition 1). In [FK] the value on the isotropic twisted conjugacy class δ is deduced from the global Proposition 2.4 of [IV] — recorded in [FK] as Theorem 0 — which asserts that $\chi_\pi(\delta) = -\chi_\pi(\delta')$.

While the proof of Proposition 2.4 of [IV] is independent of the results of [FK] (Theorems 1, 2, 3, 3′, which follow from Proposition 1), it is global, and so might lead some readers to worry that a vicious circle is created. Moreover, the proof of this global result requires heavy machinery. Here we provide a purely local proof of Proposition 1 of [FK], and consequently make the results of [FK] independent of Proposition 2.4 of [IV] (= Theorem 0 of [FK]).

Of course the conventional approach is to deduce the character computation of [FK], Proposition 1, on using the global trace formula comparison ([IV]) which is based on the fundamental lemma, proven purely locally in [V]. The novel approach of [FK] — which we complete here — is in reversing this perspective, and using the global trace formula to prove the (unstable) fundamental lemma from a purely local computation of the twisted character in a special case.

Further, an independent, direct computation of the very precise character calculation gives another assurance of the validity of the trace formula approach to the lifting project. It will be interesting to develop this approach in other lifting situations, especially since our technique is different from the well-known, standard techniques of trace formulae and dual reductive pairs. A first step in this direction was taken in our work [FZ], where the twisted — by the transpose-inverse involution — character of a representation of PGL(4) analogous to the one considered here, is computed. The situation of [FZ] is new, dealing with the exterior product of two representations of GL(2) and the structure of representations of the rank two symplectic group. Such character computations are not yet available by any other technique. However, the computations of [FZ] — although elementary — are involved, as they depend on the classification of [F] of the twisted conjugacy classes in GL(4). This is another reason for the present work, which considers the initial non trivial case of our technique — where the computations are still simple and can clarify the method. We believe that our methods, pursued in [FZ] in a more complicated case, would apply in quite general lifting situations, in conjunction with, and as an alternative to the trace formula.

Proposition 1 of [FK] asserts that if $\mathbf{1}$ is the trivial PGL(2,F)-module, $\pi = I(\mathbf{1})$ is the PGL(3,F)-module normalizedly induced from the trivial representation of the maximal parabolic subgroup (whose Levi component is GL(2,F)), and δ is a σ-regular element of PGL(3,F) with elliptic regular norm $\gamma_1 = N_1\delta$, then

$(\Delta(\delta)/\Delta_1(\gamma_1))\chi_\pi(\delta) = \kappa(\delta)$.

The proof of Proposition 1 in [FK] reduces this to the claim that the value at $s = 0$ of

$$|4u\theta|^{1/2}|u(\alpha^2-\theta)|^{-s/2}\int_{V^0}|x^2+uy^2-\theta z^2|^{3(s-1)/2}dxdydz$$

is $-\kappa(\delta)q^{-1/2}(1+q^{-1/2}+q^{-1})$ (see bottom of page 499, and Lemma 2, in [FK]).

This equality is verified in [FK], p. 499, when the quadratic form $x^2+uy^2-\theta z^2$ is anisotropic, in which case $\kappa(\delta) = -1$ and the integral converges for all s.

Here we deal with the case where the quadratic form is **isotropic**, in which case $\kappa(\delta) = 1$, the integral converges only in some half plane of s, and the value at $s = 0$ is obtained by analytic continuation.

Recall that F is a local non-archimedean field of odd residual characteristic; R denotes the (local) ring of integers of F; $\boldsymbol{\pi}$ signifies a generator of the maximal ideal of R. Denote by q the number of elements of the residue field $R/\boldsymbol{\pi}R$ of R. By $\mathbb{F}$ we mean a set of representatives in R for the finite field $R/\boldsymbol{\pi}$. The absolute value on F is normalized by $|\boldsymbol{\pi}| = q^{-1}$.

The case of interest is that where $K = F(\sqrt{\theta})$ is a quadratic extension of F, thus $\theta \in F^\times - F^{\times 2}$. Since the twisted character depends only on the twisted conjugacy class, we may assume that $|\theta|$ and $|u|$ lie in $\{1, q^{-1}\}$.

0. Lemma: *We may assume that the quadratic form $x^2+uy^2-\theta z^2$ takes one of three avatars: $x^2-\theta z^2-y^2$, $\theta \in R - R^2$; $x^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}y^2$; or $x^2-\boldsymbol{\pi}z^2-y^2$.*

Proof: (1) If K/F is unramified, then $|\theta| = 1$, thus $\theta \in R^\times - R^{\times 2}$. The norm group $N_{K/F}K^\times$ is $\pi^{2\mathbb{Z}}R^\times$. If $x^2-\theta z^2+uy^2$ represents 0 then $-u \in R^\times$. If -1 is not a square, thus $\theta = -1$, then u is -1 (get $x^2-z^2-y^2$) or $u = 1$ (get $x^2-z^2+y^2$, equivalent case). If $-1 \in R^{\times 2}$, the case of $u = \theta$ ($x^2-\theta z^2+\theta y^2 = \theta(y^2+\theta^{-1}x^2-z^2)$) is equivalent to the case of $u = -1$. So wlog $u = -1$ and the form is $x^2-\theta z^2-y^2$, $|u\theta| = 1$.

(2) If K/F is ramified, $|\theta| = q^{-1}$ and $N_{K/F}K^\times = (-\theta)^{\mathbb{Z}}R^{\times 2}$. The form $x^2-\theta z^2+uy^2$ represents zero when $-u \in R^{\times 2}$ or $-u \in -\theta R^{\times 2}$. Then the form looks like $x^2-\theta z^2+\theta y^2$ with $u = \theta$ and $|\theta u| = q^{-2}$, or $x^2-\theta z^2-y^2$ with $u = -1$ and $|\theta u| = q^{-1}$. The Lemma follows. ∎

We are interested in the value at $s = -3/2$ of the integral $I_s(u,\theta)$ of $|x^2+uy^2-\theta z^2|^s$ over the set $V^0 = V/\sim$, where

$$V = \{\mathbf{v} = (x,y,z) \in R^3; \max\{|x|,|y|,|z|\} = 1\}$$

and $\sim$ is the equivalence relation $\mathbf{v} \sim \alpha\mathbf{v}$ for $\alpha \in R^\times$.

The set V^0 is the disjoint union of the subsets

$$V_n^0 = V_n^0(u,\theta) = V_n(u,\theta)/\sim,$$

where

$$V_n = V_n(u,\theta) = \{\mathbf{v}; \max\{|x|,|y|,|z|\} = 1, |x^2+uy^2-\theta z^2| = 1/q^n\},$$

over $n \geq 0$, and of $\{\mathbf{v}; x^2+uy^2-\theta z^2 = 0\}/\sim$, a set of measure zero.

Thus we have

$$I_s(u,\theta) = \sum_{n=0}^{\infty} q^{-ns}\operatorname{Vol}(V_n^0(u,\theta)).$$

THEOREM: *The value of* $|u\theta|^{1/2}I_s(u,\theta)$ *at* $s=-3/2$ *is* $-q^{-1/2}(1+q^{-1/2}+q^{-1})$.

The problem is simply to compute the volumes

$$\operatorname{Vol}(V_n^0(u,\theta)) = \operatorname{Vol}(V_n(u,\theta))/(1-1/q) \quad (n\geq 0).$$

1. LEMMA: *When* $\theta = \boldsymbol{\pi}$ *and* $u=-1$, *thus* $|u\theta| = 1/q$, *we have*

$$\operatorname{Vol}(V_n^0) = \begin{cases} (1-1/q), & \text{if } n=0, \\ 2q^{-1}(1-1/q)+1/q^2, & \text{if } n=1, \\ 2q^{-n}(1-1/q), & \text{if } n\geq 2. \end{cases}$$

Proof: Recall that

$$V_0 = V_0(-1,\boldsymbol{\pi}) = \{(x,y,z); \max\{|x|,|y|,|z|\} = 1, |x^2-y^2-\boldsymbol{\pi}z^2| = 1\}.$$

Since $|z|\leq 1$, we have $|\boldsymbol{\pi}z^2| < 1$, and

$$1 = |x^2-y^2-\boldsymbol{\pi}z^2| = |x^2-y^2| = |x-y||x+y|.$$

Thus $|x-y| = |x+y| = 1$, and if $|x|\neq|y|$, $|x\pm y| = \max\{|x|,|y|\}$. We split V_0 into three distinct subsets, corresponding to the cases $|x|=|y|=1$; $|x|=1$, $|y|<1$; and $|x|<1$, $|y|=1$. The volume is then

$$\begin{aligned}\operatorname{Vol}(V_0) &= \int_{|z|\leq 1}\int_{|x|=1}\left[\int_{|y|=1,|x-y|=|x+y|=1}\right]dydxdz \\ &\quad + \int_{|z|\leq 1}\left[\int_{|x|=1}\int_{|y|<1} + \int_{|x|<1}\int_{|y|=1}\right]dydxdz \\ &= \int_{|x|=1}\left[\int_{|y|=1,|x-y|=|x+y|=1}\right]dydx + \frac{2}{q}\Big(1-\frac{1}{q}\Big) = \Big(1-\frac{1}{q}\Big)^2.\end{aligned}$$

To consider the V_n with $n \geq 1$, where $|x^2 - y^2 - \boldsymbol{\pi} z^2| = 1/q^n$, recall that any p-adic number a such that $|a| \leq 1$ can be written as a power series in $\boldsymbol{\pi}$:

$$a = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i = a_0 + a_1 \boldsymbol{\pi} + a_2 \boldsymbol{\pi}^2 + \cdots \quad (a_i \in \mathbb{F}).$$

In particular $|a| = 1/q^n$ implies that $a_0 = a_1 = \cdots = a_{n-1} = 0$, and $a_n \neq 0$. If

$$x = \sum_{i=0}^{\infty} x_i \boldsymbol{\pi}^i, \quad y = \sum_{i=0}^{\infty} y_i \boldsymbol{\pi}^i, \quad z = \sum_{i=0}^{\infty} z_i \boldsymbol{\pi}^i \quad (x_i, y_i, z_i \in \mathbb{F}),$$

then

$$x^2 = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i, \quad y^2 = \sum_{i=0}^{\infty} b_i \boldsymbol{\pi}^i, \quad z^2 = \sum_{i=0}^{\infty} c_i \boldsymbol{\pi}^i,$$

where

$$a_i = \sum_{j=0}^{i} x_j x_{i-j}, \quad b_i = \sum_{j=0}^{i} y_j y_{i-j}, \quad c_i = \sum_{j=0}^{i} z_j z_{i-j} \quad (a_i, b_i, c_i \in \mathbb{F}).$$

We have

$$x^2 - y^2 - \boldsymbol{\pi} z^2 = \sum_{i=0}^{\infty} f_i \boldsymbol{\pi}^i \quad (f_i \in \mathbb{F}),$$

where $f_0 = a_0 - b_0$, $f_i = a_i - b_i - c_{i-1}$ $(i \geq 1)$. Since $|x^2 - y^2 - \boldsymbol{\pi} z^2| = 1/q^n$, we have that $f_0 = f_1 = \cdots = f_{n-1} = 0$, and $f_n \neq 0$. Thus we obtain the relations (for a, b, c in the set $\mathbb{F}$, which (modulo $\boldsymbol{\pi}$) is the field $R/\boldsymbol{\pi}$):

$$a_0 - b_0 = 0, \quad a_i - b_i - c_{i-1} = 0 \quad (i = 1, \ldots, n-1), \quad a_n - b_n - c_{n-1} \neq 0.$$

Recall that together with $\max\{|x|, |y|, |z|\} = 1$, these relations define the set V_n.

To compute the volume of V_n we integrate in the order: $\ldots dydzdx$. From $a_0 - b_0 = 0$ it follows that $y_0 = \pm x_0$, and from $a_i - b_i - c_{i-1}$ $(i \geq 1)$ it follows that

$$2y_0 y_i = a_i - c_{i-1} - \sum_{j=1}^{i-1} y_j y_{i-j},$$

where in the case of $i = 1$ the sum over j is empty.

Let $n \geq 2$. When $i = 1$ we have $2x_0 x_1 - 2y_0 y_1 - z_0^2 = 0$. So if $x_0 = 0$ (in $R/\boldsymbol{\pi}$, i.e., $|x| < 1$), it follows that $y_0 = 0$ and $z_0 = 0$ (i.e., $|y| < 1$, $|z| < 1$). This contradicts the fact that $\max\{|x|, |y|, |z|\} = 1$. Thus $|x| = 1$. In this case $y_0 \neq 0$ and (for $n \geq 2$) we have

$$\mathrm{Vol}(V_n) = \int_{|x|=1} \int_{|z| \leq 1} \left[\int dy \right] dzdx,$$

where the variable y is such that once written as $y = y_0 + y_1\boldsymbol{\pi} + y_2\boldsymbol{\pi}^2 + \cdots$, it has to satisfy $y_0 = \pm x_0$, and y_i $(i = 1, \ldots, n-1)$ is defined uniquely from $a_i - b_i - c_{i-1} = 0$, and $y_n \neq$ some value defined by $a_n - b_n - c_{n-1} \neq 0$. Thus when $n \geq 2$,

$$\mathrm{Vol}(V_n) = \frac{2}{q}\Big(\frac{1}{q}\Big)^{n-1}\Big(1 - \frac{1}{q}\Big)^2 = \frac{2}{q^n}\Big(1 - \frac{1}{q}\Big)^2.$$

Let $n = 1$. When $i = 1$ we have $2x_0x_1 - 2y_0y_1 - z_0^2 \neq 0$. So if $x_0 = 0$ (i.e., $|x| < 1$), it follows that $y_0 = 0$ and $z_0 \neq 0$ (i.e., we have an additional contribution from $|x| < 1$, $|y| < 1$, $|z| = 1$). Thus,

$$\mathrm{Vol}(V_1) = \frac{2}{q}\Big(1 - \frac{1}{q}\Big)^2 + \frac{1}{q^2}\Big(1 - \frac{1}{q}\Big).$$

The Lemma follows. ∎

2. Lemma: *When u and θ equal $\boldsymbol{\pi}$, thus $|u\theta| = 1/q^2$, we have*

$$\mathrm{Vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-1}(1 - 1/q), & \text{if } n = 1, \\ 2q^{-n}(1 - 1/q), & \text{if } n \geq 2. \end{cases}$$

Proof: To compute $\mathrm{Vol}(V_0)$, recall that

$$V_0 = \{(x, y, z); \max\{|x|, |y|, |z|\} = 1, |x^2 + \boldsymbol{\pi}(y^2 - z^2)| = 1\}.$$

Since $|y| \leq 1$, $|z| \leq 1$, we have $|x^2 + \boldsymbol{\pi}(y^2 - z^2)| = |x^2| = 1$, and so

$$\mathrm{Vol}(V_0) = \int_{|z|\leq 1}\int_{|y|\leq 1}\int_{|x|=1} dxdydz = 1 - \frac{1}{q}.$$

To compute $\mathrm{Vol}(V_n)$, $n \geq 1$, recall that

$$V_n = \{(x, y, z); \max\{|x|, |y|, |z|\} = 1, |x^2 + \boldsymbol{\pi}(y^2 - z^2)| = 1/q^n\}.$$

Following the notations of Lemma 1 we write

$$x^2 + \boldsymbol{\pi}(y^2 - z^2) = \sum_{i=0}^{\infty} f_i\boldsymbol{\pi}^i \quad (f_i \in \mathbb{F}),$$

where $f_0 = a_0$ and $f_i = a_i + b_{i-1} - c_{i-1}$ $(i \geq 1)$. The condition which defines V_n is that $f_0 = f_1 = \cdots = f_{n-1} = 0$ and $f_n \neq 0$. The equation $f_0 = 0$ implies that $x_0 = 0$ (i.e., $|x| < 1$). We arrange the order of integration to be $\ldots dydzdx$.

When $n \geq 2$, since $x_0 = 0$, $f_1 = 0$ implies that $y_0^2 - z_0^2 = 0$. Using $\max\{|x|, |y|, |z|\} = 1$ we conclude that $y_0 = \pm z_0 \neq 0$ (i.e., $|z| = 1$, $|z^2 - y^2| < 1$). Thus we have

$$\mathrm{Vol}(V_n) = \int_{|x|<1} \int_{|z|=1} \left[\int dy \right] dz dx$$

where the variable y is such that once written as $y = y_0 + y_1\pi + y_2\pi^2 + \cdots$, it has to satisfy $y_0 = \pm z_0$, and y_i $(i = 1, ..., n-2)$ is defined uniquely from $a_i + b_{i-1} - c_{i-1} = 0$, and $y_{n-1} \neq$ some value defined by $a_n + b_{n-1} - c_{n-1} \neq 0$. Thus when $n \geq 2$,

$$\mathrm{Vol}(V_n) = \frac{1}{q}\frac{2}{q}\Big(\frac{1}{q}\Big)^{n-2}\Big(1 - \frac{1}{q}\Big)^2 = \frac{2}{q^n}\Big(1 - \frac{1}{q}\Big)^2.$$

When $n = 1$ we have $f_0 = 0$, $f_1 \neq 0$. These amount to $x_0 = 0$, $y_0 \neq \pm z_0$. Separating the two cases $z_0 = 0$ and $z_0 \neq 0$, we obtain

$$\begin{aligned}\mathrm{Vol}(V_1) &= \int_{|x|<1} \int_{|z|<1} \int_{|y|=1} dy dz dx + \int_{|x|<1} \int_{|z|=1} \int_{|y^2-z^2|=1} dy dz dx \\ &= \frac{1}{q^2}\Big(1 - \frac{1}{q}\Big) + \frac{1}{q}\Big(1 - \frac{1}{q}\Big)\Big(1 - \frac{2}{q}\Big) = \frac{1}{q}\Big(1 - \frac{1}{q}\Big)^2.\end{aligned}$$

The Lemma follows. ∎

3. LEMMA: *When K/F is unramified, thus $|u\theta| = 1$, we have*

$$\mathrm{Vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-n}(1 - 1/q)(1 + 1/q), & \text{if } n \geq 1. \end{cases}$$

Proof: First we compute $\mathrm{Vol}(V_0)$. Recall that

$$V_0 = \{(x, y, z); \max\{|x|, |y|, |z|\} = 1, |x^2 - y^2 - \theta z^2| = 1\}.$$

Since $|x^2 - y^2 - \theta z^2| \leq \max\{|x|, |y|, |z|\}$,

$$V_0 = \{(x, y, z) \in R^3; |x^2 - y^2 - \theta z^2| = 1\}.$$

Making the change of variables $u = x + y$, $v = x - y$, we obtain

$$V_0 = \{(u, v, z) \in R^3; |uv - \theta z^2| = 1\}.$$

Assume that $|uv| < 1$. Since $|uv - \theta z^2| = 1$, it follows that $|z| = 1$. The contribution from the set $|uv| < 1$ is

$$\begin{aligned}&\int_{|z|=1} \left[\int_{|u|<1} \int_{|v|\leq 1} + \int_{|u|=1} \int_{|v|<1} \right] du dv dz \\ &= \Big(1 - \frac{1}{q}\Big)\Big(\frac{1}{q} + \Big(1 - \frac{1}{q}\Big)\frac{1}{q}\Big) = \frac{1}{q}\Big(1 - \frac{1}{q}\Big)\Big(2 - \frac{1}{q}\Big).\end{aligned}$$

Assume that $|uv| = 1$, i.e., $|u| = |v| = 1$. We arrange the order of integration as $dudvdz$. If $|z| < 1$ then $|uv - \theta z^2| = |uv| = 1$. If $|z| = 1$ we introduce $U(v,z) = \{u;\ |u| = 1,\ |uv - \theta z^2| = 1\}$, a set of volume $1 - 2/q$, and note that the contribution from the set $|uv| = 1$ is

$$\int_{|z|<1}\int_{|v|=1}\int_{|u|=1} dudvdz + \int_{|z|=1}\int_{|v|=1}\int_{U(v,z)} dudvdz.$$

The sum of the two integrals is

$$\frac{1}{q}\Big(1-\frac{1}{q}\Big)^2 + \Big(1-\frac{1}{q}\Big)^2\Big(1-\frac{2}{q}\Big) = \Big(1-\frac{1}{q}\Big)^3.$$

Adding the contributions from $|uv| < 1$ and $|uv| = 1$ we then obtain

$$\mathrm{Vol}(V_0) = \frac{1}{q}\Big(1-\frac{1}{q}\Big)\Big(2-\frac{1}{q}\Big) + \Big(1-\frac{1}{q}\Big)^3 = 1 - \frac{1}{q}.$$

Next we compute $\mathrm{Vol}(V_n)$, $n \geq 1$. Recall that

$$V_n = \{(x,y,z); \max\{|x|,|y|,|z|\} = 1, |x^2 - y^2 - \theta z^2| = 1/q^n\}.$$

Making the change of variables $u = x + y$, $v = x - y$, we obtain

$$V_n = \{(u,v,z); \max\{|u+v|,|u-v|,|z|\} = 1, |uv - \theta z^2| = 1/q^n\}.$$

Since the set $\{v = 0\}$ is of measure zero, we assume that $v \neq 0$. Then $|uv - \theta z^2| = 1/q^n$ implies that $u = \theta z^2 v^{-1} + t v^{-1} \pi^n$, where $|t| = 1$. There are two cases.

Assume that $|v| = 1$. Note that if $|z| = 1$, then $\max\{|u+v|,|u-v|,|z|\} = 1$ is satisfied, and if $|z| < 1$, then (recall that $n \geq 1$)

$$|u| = |\theta z^2 v^{-1} + t v^{-1} \pi^n| \leq \max\{|z^2|, q^{-n}\} < 1,$$

and $|u+v| = |v| = 1$. So $|v| = 1$ implies $\max\{|u+v|,|u-v|,|z|\} = 1$. Further, since $|v| = 1$, we have $du = q^{-n}dt$. Thus the contribution from the set with $|v| = 1$ is

$$\int_{|z|\leq 1}\int_{|v|=1}\int_{|uv-\theta z^2|=1/q^n} dudvdz = \int_{|z|\leq 1}\int_{|v|=1}\int_{|t|=1} \frac{dt}{q^n} dvdz = \frac{1}{q^n}\Big(1-\frac{1}{q}\Big)^2.$$

Assume that $|v| < 1$. Note that if $|z| = 1$, since $|u| \leq 1$ we have $q^{-n} = |uv - \theta z^2| = |\theta z^2| = 1$, a contradiction. Thus $|z| < 1$, and in order to satisfy $\max\{|u+v|,|u-v|,|z|\} = 1$, we should have $|u| = 1$. The contribution from the set with $|v| < 1$ is

$$\int_{|z|<1}\int_{|u|=1}\int_{|uv-\theta z^2|=1/q^n} dvdudz.$$

We write $v = \theta z^2 u^{-1} + t u^{-1} \boldsymbol{\pi}^n$, where $|t| = 1$, and $dv = q^{-n} dt$. The integral equals

$$\int_{|z|<1} \int_{|u|=1} \int_{|t|=1} \frac{dt}{q^n} du dz = \frac{1}{q} \frac{1}{q^n} \Big(1 - \frac{1}{q}\Big)^2.$$

Adding the contributions from $|v| = 1$ and $|v| < 1$ we obtain

$$\mathrm{Vol}(V_n) = \frac{1}{q^n} \Big(1 - \frac{1}{q}\Big)^2 + \frac{1}{q} \frac{1}{q^n} \Big(1 - \frac{1}{q}\Big)^2 = \frac{1}{q^n} \Big(1 - \frac{1}{q}\Big)^2 \Big(1 + \frac{1}{q}\Big).$$

The Lemma follows. ∎

This completes the proof of the theorem, so that we have provided a purely local proof of (the character relation of) Proposition 1 of [FK]. We believe that analogous computations can be carried out in other lifting situations, to provide direct and local computations of twisted characters. As noted in the introduction, a step in this direction is taken in [FZ].

References

[IV] Y. Flicker, *On the symmetric square. Applications of a trace formula*, Transactions of the American Mathematical Society **330** (1992), 125–152.

[V] Y. Flicker, *On the symmetric-square. The fundamental lemma*, Pacific Journal of Mathematics **175** (1996), 507–526.

[F] Y. Flicker, *Matching of orbital integrals on GL(4) and GSp(2)*, Memoirs of the American Mathematical Society **137** (1999), 1–114.

[FK] Y. Flicker and D. Kazhdan, *On the symmetric-square: Unstable local transfer*, Inventiones Mathematicae **91** (1988), 493–504.

[FZ] Y. Flicker and D. Zinoviev, *Twisted character of a small representation of PGL(4)*, preprint.